Præcipuarum Functionum Trigonometricarum per Analysin Infinitorum Explicatio.

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$du = \frac{dx}{1+x^2}$; e cotangente x per integrationem formulæ $du = -\frac{dx}{1+x^2}$; cet. Negandum non est hanc methodum arcum e functionibus suis trigonometricis per integrationem directam quærendi simpliciorem videri posse superioribus, quæ tamen principiis suis, ubertate et multimodo usu excellunt.

Schol. 4. Præter illa, quæ inter arcum ejusque lineas trigonometricas jam ostendimus functionum vincula, alia quoque sunt, quæ etsi ad singulorum computationem nihil valeant, vim tamen mirificam habent ad reliquas quæstiones goniometricas solvendas, quales sunt, quæ in lineis arcuum multiplorum, in dignitatibus sinuum, cosinuum, cet, in summis et differentiis horum definiendis versantur. Sistunt se nova hæc arcus et linearum foedera ope quantitatum exponentialium et logarithmicarum. Duæ autem sunt inprimis, quibus illa investigentur, viæ; vel per integrationem solius formulæ $du\sqrt{-1} = \frac{dx}{\sqrt{x^2-1}}$, vel per ea, quæ e præcedentibus problematis jam nota sunt. Ab illa igitur, quæ minimum mutuatur, ordiendum.

Sit arcus u, sinus ejus x, erit

$$du\sqrt{-1} = \frac{dx}{(x^2-1)^{\frac{1}{2}}} = \frac{dx}{X}, \text{ brevitatis caussa, } = dx\,\frac{\left(1+\frac{x}{X}\right)}{X\left(1+\frac{x}{X}\right)}$$

$$= \frac{dx + \frac{dx}{(x^2-1)^{\frac{1}{2}}}}{x+(x^2-1)^{\frac{1}{2}}} = d\log\left(x+(x^2-1)^{\frac{1}{2}}\right);$$ quo probato, habebitur

$$\int du\sqrt{-1} = u\sqrt{-1} = \log\left(Cx + C(x^2-1)^{\frac{1}{2}}\right);$$ et posito basi systematis naturalis $= e$, $e^{u\sqrt{-1}} = Cx + C(x^2-1)^{\frac{1}{2}}$. Hinc

$$e^{2u\sqrt{-1}} - 2Cxe^{u\sqrt{-1}} + C^2x^2 = C^2x^2 - C^2; \text{ sive } 2Cxe^{u\sqrt{-1}} = e^{2u\sqrt{-1}} + C^2.$$

Facto autem x, et cum illo $u, = o$, adparebit mox esse $C^2 = -1$; sive $C = \sqrt{-1}$; qui valor constantis, rite insertus, dabit completum integrale

$$x = \frac{e^{2u\sqrt{-1}} - 1}{e^{u\sqrt{-1}} \times 2\sqrt{-1}} = \frac{e^{u\sqrt{-1}} - e^{-u\sqrt{-1}}}{2\sqrt{-1}} = \sin u.$$

Sit $\cos u = y$, erit

$$y = \left(1 - \frac{e^{2u\sqrt{-1}} - 2 + e^{-2u\sqrt{-1}}}{-4}\right)^{\frac{1}{2}} = \left(\frac{2 + e^{2u\sqrt{-1}} + e^{-2u\sqrt{-1}}}{4}\right)^{\frac{1}{2}}$$

$$= \frac{e^{u\sqrt{-1}} + e^{-u\sqrt{-1}}}{2}.$$

Altera via, quæ per jam inventa ducat, hæc esto:

$$x\sqrt{-1} = \left(u - \frac{u^3}{1.2.3} + \frac{u^5}{1.2.3.4.5} - \ldots\ldots\right)\sqrt{-1}\text{; et}$$

$$y = 1 - \frac{u^2}{1.2} + \frac{u^4}{1.2.3.4} - \frac{u^6}{1.2.3.4.5.6} + \ldots\ldots;$$

e theoria vero logarithmorum

$$e^{u\sqrt{-1}} = 1 + u\sqrt{-1} - \frac{u^2}{1.2} - \frac{u^3\sqrt{-1}}{1.2.3} + \frac{u^4}{1.2.3.4} + \frac{u^5\sqrt{-1}}{1.2.3.4.5} - \frac{u^6}{1.2.3.4.5.6} - \ldots;$$

quo fiet $e^{u\sqrt{-1}} = y + x\sqrt{-1} = \cos u + \sin u\sqrt{-1}$.

Eadem ratione habebimus

$e^{-u\sqrt{-1}} = y - x\sqrt{-1}$; unde, exterminatione facta, erit

$$x = \frac{e^{u\sqrt{-1}} - e^{-u\sqrt{-1}}}{2\sqrt{-1}};\ y = \frac{e^{u\sqrt{-1}} + e^{-u\sqrt{-1}}}{2}\text{; ut supra.}$$

Denique prodibit

$$e^{2u\sqrt{-1}} = \frac{1 + \sqrt{-1}.\frac{x}{y}}{1 - \sqrt{-1}.\frac{x}{y}} = \frac{1 + \sqrt{-1}.\tang u}{1 - \sqrt{-1}.\tang u}$$

Probl. 16. *Datis sinibus et cosinibus invenire sinum summæ et sinum differentiæ arcuum.*

Primum hoc et fundamentale problematum trigonometricorum eleganter solvetur iis, quæ in præcedente scholio exposita fuere.

Sint u, v anguli, quorum sinus et cosinus dati. Ponatur brevitatis causa, $u\sqrt{-1}=q$; $v\sqrt{-1}=r$; $\sqrt{-1}=i$, habebimus mox

$$\sin u \cos v=\left(\frac{e^q-e^{-q}}{2i}\right)\left(\frac{e^r+e^{-r}}{2}\right)=\frac{e^{q+r}-e^{r-q}+e^{q-r}-e^{-q-r}}{4i}.$$

$$\cos u \sin u=\left(\frac{e^q+e^{-q}}{2}\right)\left(\frac{e^r-e^{-r}}{2i}\right)=\frac{e^{q+r}+e^{r-q}-e^{q-r}-e^{-q-r}}{4i}.$$

Hinc $\sin u \cos v+\cos u \sin v=\dfrac{2e^{q+r}-2e^{-q-r}}{4i}=\dfrac{e^{(u+v)\sqrt{-1}}-e^{-(u+v)\sqrt{-1}}}{2\sqrt{-1}}$; sive

α) $\sin u \cos v+\cos u \sin v=\sin(u+v)$.

Pari modo invenietur

$$\sin u \cos v-\cos u \sin v=\frac{2e^{q-r}-2e^{r-q}}{4i}=\frac{e^{(u-v)\sqrt{-1}}-e^{-(u-v)\sqrt{-1}}}{2\sqrt{-1}};\ \text{sive}$$

β) $\sin u \cos v-\cos u \sin v=\sin(u-v)$

Perpendenti formulas: $\sin u=\dfrac{e^q-e^{-q}}{2i}$; $\cos u=\dfrac{e^q+e^{-q}}{2}$; per solam integrationem form. $du\sqrt{-1}=-\dfrac{dx}{(x^2-1)}$ exstitisse, vel hoc modo clarum erit huic problemati fundamentali, analyticis subsidiis munito, ut solvatur, nullam aliam constructionem geometricam requiri, quam quæ formulæ: $du=\dfrac{dx}{\sqrt{1-x^2}}$ subjecta est.

Probl. 17. *Iisdem datis, invenire cosinum summæ et cosinum differentiæ arcuum.*

Erit $\cos u \cos v=\left(\dfrac{e^q+e^{-q}}{2}\right)\left(\dfrac{e^r+e^{-r}}{2}\right)=\dfrac{e^{q+r}+e^{r-q}+e^{q-r}+e^{-q-r}}{4}$;

$$\sin u \sin v=\left(\frac{e^q-e^{-q}}{2i}\right)\left(\frac{e^r-e^{-r}}{2i}\right)=\frac{e^{q+r}-e^{r-q}-e^{q-r}+e^{-q-r}}{-4}.$$

Hinc $\cos u \cos v-\sin u \sin v=\dfrac{2e^{q+r}+2e^{-(q+r)}}{4}=\dfrac{e^{(u+v)\sqrt{-1}}+e^{-(u+v)\sqrt{-1}}}{2}$; sive

α) $\cos u \cos v-\sin u \sin v=\cos(u+v)$

Similiter habebimus

$$\cos u \cos v + \sin u \sin v = \frac{2e^{-(q-r)} + 2e^{-(q-r)}}{4} = \frac{e^{(u-v)\sqrt{-1}} + e^{-(u-v)\sqrt{-1}}}{2}; \text{ sive}$$

$$\beta)\ \cos u \cos v + \sin u \sin v = \cos(u-v).$$

Probl. 18. *Dato sinu et cosinu arcus simplicis invenire sinum et cosinum arcus multipli* (nu).

Cum sit $\sin u = \frac{e^q - e^{-q}}{2i}$ et $\cos u = \frac{e^q + e^{-q}}{2}$;

sequitur esse

$\cos u + i \sin u = e^q = e^{u\sqrt{-1}}$, atque $\cos u - i \sin u = e^{-q} = e^{-u\sqrt{-1}}$.

Habebimus porro:

$(e^{u\sqrt{-1}})^n = e^{nu\sqrt{-1}} = (\cos u + i \sin u)^n$, et $(e^{-u\sqrt{-1}})^n = e^{-nu\sqrt{-1}} = (\cos u - i \sin u)^n$;

unde, cum sit $\sin(nu) = \frac{e^{nu\sqrt{-1}} - e^{-nu\sqrt{-1}}}{2\sqrt{-1}}$, erit

$$\alpha) \sin(nu) = \frac{(\cos u + i \sin u)^n - (\cos u - i \sin u)^n}{2i} = \frac{(\cos u + \sin u\sqrt{-1})^n - (\cos u - \sin u\sqrt{-1})^n}{2\sqrt{-1}}.$$

Simili ratione invenietur:

$$\beta) \cos nu = \frac{(\cos u + \sin u\sqrt{-1})^n + (\cos u - \sin u\sqrt{-1})^n}{2}.$$

Nihil officiet computaturo imaginaria. Qui enim, ceteris invicem compensatis, in numeratore formulæ α) restant termini, iidem illam, primi tantum gradus, factorem habent. Quæ vero illa in numeratore formulæ β afficiuntur quantitates, subtractione tolluntur omnes. Præstat hic, quod sæpe alias, φάσμα illud analyticum. Concinniore formula, dum adest, dumque opus est, omnia detinet; dum non amplius opus, oculis se nostris, haud raro mirantibus, subducit.

Probl. 19. *Dignitates sinus, vel cosinus, per easdem, sed primi gradus, functiones arcus simplicis et multipli exprimere.*

Sit n exponens dignitatis, ad quam elevatus est sinus u; habebimus

$$\sin^n u = \left(\frac{e^q - e^{-q}}{2i}\right)^n = \frac{1}{2i^n}(e^q - e^{-q})^n$$

Erit autem $(e^q - e^{-q})^n = e^{nq} - ne^{(n-2)q} + n.\frac{(n-1)}{1.2}e^{(n-4)q} - n.\frac{(n-1)(n-2)}{1.2.3}e^{(n-6)q}$

$+ n\,\frac{(n-1)(n-2)(n-3)}{1.2.3.4.}e^{(n-8)q} - \ldots\ldots = Q,$

quocirca $\alpha)\; sin^n u = \frac{1}{(2i)^n} \times Q = \frac{Q}{(2\sqrt{-1})^n}$

Neque hic, etsiamsi n fuerit numerus impar, impedietur imaginaria, quisquis ortum formulæ et vim perspexerit. Est autem hæc ejus vis, ut binis numeratoris terminis apte junctis, si n fuerit numerus impar, secundum formulam $sin\ u = \frac{e^q - e^{-q}}{2i}$, sinus tantum, simplicis arcus et multipli, sin vero fuerit numerus par, secundum formulam $\frac{e^q + e^{-q}}{2}$ cosinus multipli, cum numero, pro n constante, exhibeat.

Qualis futurus sit valor $cos^2 u = \frac{1}{2^n}(e^q + e^{-q})^n$, e valore α) satis patet. Erit nempe $cos^n u = \frac{1}{2^n} \times Q'$; ubi Q' continebitur iisdem terminis quibus Q; hoc solum discrimine, ut negativa signa in positiva abeant, quo fient termini omnes positivi. Perspicuum quoque est hanc formulam, ita adornatam, nullos sinus, sed solos cosinus multipli arcus exhibere, junctos vel cum constante, vel cum cosinu arcus simplicis, prout n numerus vel par, vel impar fuerit.

Schol. Qvamvis solutiones problematum 18 et 19 valde commodas formulas adtulerint, tamen, ut in 18:mo, neque lex ipsa, qua quæsita se excipiunt, satis explicita videatur, et, ut in ambobus, aliena theoria, logarithmica puta, in auxilium vocata fuit. Quærantur igitur nunc, ut omnibus satisfiat votis, primum sinus et cosinus arcus multipli (nu) e solis dignitatibus sinus et cosinus simplicis u, numeroque n, rationem multipli ad simplicem indicante.

Sit $sin(nu) = (Ax + Ax^3 + Cx^5 + Dx^7 + \ldots)\ \alpha)$, in quam seriem, ne $\pm\, nu$ valorem instabilem redderet, non nisi impares dignitates admisimus; erit prima differentiatione

$n\,cos(nu)du=(A+3Bx^2+5Cx^4+7Dx^6)du(1-x^2)^{\frac{1}{2}}$; sive

$n\,cos(nu)=(n+3Bx^2+5Cx^4+7Dx^6)(1-x^2)^{\frac{1}{2}}$; quia $A=n$; posito $x=0$;

atque secunda

$-n^2 sin\,nu\,du=(6Bx+20Cx^3+42Dx^5)(1-x^2)du-(n+3Bx^2+5Cx^4+7Dx^6)xdu$;

$-n^2 sin(nu)=6Bx+20Cx^3+42Dx^5-6Bx^3-20Cx^5-42Dx^7-nx-3Bx^3-5Cx^5-7Dx^7$;

$$\text{sive } 0=\left.\begin{matrix}+n^3\\+6B\\-n\end{matrix}\right\}x\;\left.\begin{matrix}+n^2B\\-9B\\+20C\end{matrix}\right\}x^3\;\left.\begin{matrix}+n^2C\\-25C\\+42D\end{matrix}\right\}x^5\;\begin{matrix}+n^2D\\-49D\\+\dots\end{matrix}$$

Hinc $6B=n-n^3$; sive $B=-\frac{(n^2-1)n}{1.2.3}$; $C=-B\frac{(n^2-3^2)}{4.5}=\frac{(n^2-1)(n^2-3^2)}{1.2.3.4.5}$;

$D=-\frac{(n^2-5^2)}{6.7}C=-n\frac{(n^2-1)(n^2-3^2)(n^2-5^2)}{1.2.3.4.5.6.7}$.

Erit igitur α) $sin(nu)=n\,sin\,u-n\frac{(n^2-1^2)}{1.2.3}sin^3u+n\frac{(n^2-1^2)(n^2-3^2)}{1.2.3.4.5}sin^5u$

$-n\frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{1.2.3.4.5.6.7}sin^7u$

Sit porro $cos(nu)=1+Ax^2+Bx^4+Cx^6+\dots\dots$ nullis, ut oportet, dignitatibus imparibus admissis.

Prima differentiatione habebimus

$-n\,sin(nu)=(2Ax+4Bx^3+6Cx^5+\dots)(1-x^2)^{\frac{1}{2}}$;

secunda

$-n^2cos\,nu=(2A+12Bx^2+30Cx^4+\dots)(1-x^2)-(2Ax+4Bx^3+6Cx^5)x$; sive

$0=n^2+n^2Ax^2+n^2Bx^4+n^2Cx^6+2A+12Bx^2+30Cx^4-2Ax^2-12Bx^4-30Cx^6-2Ax^2-4Bx^4-6Cx^6$

$$\text{sive } 0=\begin{matrix}+n^2\\+2A\\ \end{matrix}\;\left.\begin{matrix}+n^2A\\+12B\\-4A\end{matrix}\right\}x^2\;\left.\begin{matrix}+n^2B\\+30C\\-16B\end{matrix}\right\}x^4\;\left.\begin{matrix}+n^2C\\+30C\\-\dots\end{matrix}\right\}x^6+\dots$$

Hinc $A=-\frac{n^2}{2}$; $12B=-n^2(A+4A)=n^2\frac{(n^2-2^2)}{1.2.}$; sive $B=n^2\frac{(n^2-2^2)}{1.2.34}$;

$30C=-n^2\frac{(n^2-2^2)(n^2-2^4)}{1.2.3.4}$; sive $C=-n^2\frac{(n^2-2^2)(n^2-4^2)}{1.2.3.4.5.6}$; ex quibus omnibus lex seriei perspicua.

Erit scilicet

$$\beta)\ \cos(nu)=1-\frac{n^2}{1.2}\sin^2u+n^2\frac{(n^2-2^2)}{1.2.3.4}\sin^4u-n^2\frac{(n^2-2^2)(n^2-2^4)}{1.2.3.4.5.6}\sin^6u+\ldots$$

Ne α) pro numero *pari* n, β) pro *impari* eodem, infinitum valorem edat, habebimus, differentiando α),

$$ndu\cos(nu)=n\cos u du-3n\frac{(n^2-1)}{1.2.3}\sin^2u\cos u du+5n\frac{(n^2-1)(n^2-3^2)}{1.2.3.4.5}\sin^4u\cos u du-\ldots.$$

$$\text{sive } \gamma)\cos(nu)=\cos u\left(1-\frac{(n^2-1)}{1.2.}\sin^2u+\frac{(n^2-1)(n^2-3^2)\sin^4u}{1.2.3.4}-\frac{(n^2-1)(n^2-3^2)(n^2-5^2)\sin^6u}{1.2.3.4.5.6}+\ldots\right)$$

atque differentiando β),

$$-n\,du\sin(nu)=-\frac{2n^2\sin u\cos u\,du}{1.2.}+\frac{4n^2(n^2-2^2)\sin^3u\cos u\,du}{1.2.3.4}$$

$$-\frac{6n^2(n^2-2^2)(n^2-4^2)\sin^5u\cos u\,du}{1.2.3.4.5.6}+\ldots.$$

$$\text{sive } \delta)\sin(nu)=\cos u\left(n\sin u-n\frac{(n^2-2^2)\sin^3u}{1.2.3}+n\frac{(n^2-2^2)(n^2-4^2)\sin^5u}{1.2.3.4.5}+\ldots\right)$$

Si igitur n fuerit numerus impar, sinus (nu) quærendus est ex α), et $\cos(nu)$ e γ); *sin* par, $\sin(nu)$ e δ) et $\cos(nu)$ e β); quo series pro dato n ubique rite abrumpatur.

Cum e modo inventis æquationibus $(\beta+\gamma)$, factis nonnullis permutationibus cosinus cujusvis arcus multipli eo reduci possit, ut, præter constantem, meris dignitatibus cosinus simplicis exprimatur, hac lege exponentium decrescentibus: (n); $(n-2)$: $(n-4)$; $(n-6)$; cet; utamur hac ansa effingendi seriem coefficientium indeterminatorum, qua cosinus arcus simplicis, ad dignitatem n elevatus, cosinibus arcuum multiplorum ordine n; $(n-2)$; $(n-4)$; $(n-6)$; decresentium exprimatur; verbo ponamus:

$$\cos^n u=A\cos nu+B\cos(n-2)u+C\cos(n-4)u+D\cos(n-6)u+E\cos(n-8)u+\ldots$$

Differentiando erit $n\cos^{n-1}\sin u=nA\sin nu+(n-2)B\sin(n-2)u$

$+(n-4)C\sin(n-4)u+(n-6)D\sin(n-6)u+(n-8)E\sin(n-8)u+\ldots.$ sive

$$0=n\sin u(A\cos nu+B\cos(n-2)u+C\cos(n-4)u+D\cos(n-6)u+E\cos(n-8)u+\ldots)$$

$$-\cos u\Big(nA\sin(nu)+(n-2)B\sin(n-2)u+(n-4)C\sin(n-4)u+(n-6)D\sin(n-)u$$

$$+(n-8)E\sin(n-8)u+\ldots\Big)=An\times\frac{\sin(n+1)u-\sin(n-1)u}{2}$$

$$+Bn\times\frac{\sin(n-1)u-\sin(n-3)u}{2}+Cn\times\frac{\sin(n-3)u-\sin(n-5)u}{2}$$

$$+Dn\times\frac{\sin(n-5)u-\sin(n-7)u}{2}+En\times\frac{\sin(n-7)u-\sin(n-9)u}{2}+\dots$$

$$-An\times\frac{\sin(n+1)u+\sin(n-1)u}{2}-B(n-2)\times\frac{\sin(n-1)u+\sin(n-3)u}{2}$$

$$-C(n-4)\times\frac{\sin(n-3)u+\sin(n-5)u}{2}-D(n-6)\times\frac{\sin(n-5)u+\sin(n-7)u}{2}$$

$$-E(n-8)\times\frac{\sin(n-7)u+\sin(n-9)u}{2}-\dots$$

Debita igitur additione et substractione facta, obtinebitur

$$0=-An\sin(n-1)u-Bn\sin(n-3)u-Cn\sin(n-5)u-Dn\sin(n-7)u-En\sin(n-9)u-\dots$$
$$+B\sin(n-1)u+B\sin(n-3)u+2C\sin(n-3)u+2C\sin(n-5)u+3D\sin(n-5)$$
$$+3D\sin(n-7)u+4E\sin(n-7)u+4E\sin(n-9)u+\dots$$

$$=\left.\begin{matrix}B\\-An\\ \end{matrix}\right\}\sin(n-1)u+\left.\begin{matrix}-Bn\\+B\\+2C\end{matrix}\right\}\sin(n-3)u\left.\begin{matrix}-Cn\\+2C\\+3D\end{matrix}\right\}\sin(n-5)u\left.\begin{matrix}-Dn\\+3D\\+4E\end{matrix}\right\}\sin(n-7)u+$$

Hinc $B=nA;C=B\frac{(n-1)}{2};=An\frac{(n-1)}{1.2};D=C\frac{(n-2)}{3}=An\frac{(n-1)(n-2)}{1.2.3};$

$E=D\times\frac{n-3}{4}=An\frac{(n-1)(n-2)(n-3)}{1.2..34.}$; quibus fiet

$$\cos^n u=A\Big(\cos nu+\frac{n}{1}\cos(n-2)u+n\frac{(n-1)}{1.2}\cos(n-4)u+n\frac{(n-1)(n-2)}{1.2.3}\cos(n-6)u$$
$$+n\frac{(n-1)(n-2)(n-3)}{1.2.3.4}\cos(n-8)u+\dots\Big)=Q,$$ sive

$\cos^n u=\frac{1}{2^n}\times Q$: quia posito $x=o$, erit $1=A\times(1+1)^n=A\times2^n$, unde $A=\frac{1}{2^n}$.

Habebimus igitur $\text{Cos}^n u=\frac{A}{2^n}\Big(\cos nu+\frac{n}{1}\cos(n-2)u+n\frac{(n-1)}{2}\cos(n=4)u$

$$+n\frac{(n-1)(n-2)}{1.2.3}\cos(n-6)u+n\frac{(n-1)(n-2)(n-3)}{1.2.3.4}\cos(n-8)u+\dots\Big)$$

Quia vero in hac serie, antequam ad coefficientem, cifræ æqualem, perventum fuerit, cosinus negativorum angulorum totidem atque cosinus positivorum occurrent, cum hisce, vel signis, æquales, sequitur, si cum cosinibus positivorum angulorum series abrumpenda sit, hanc fore æquationem quæsitam $\cos^n u=\frac{1}{2^n}\times2Q'=\frac{1}{2^{n-1}}Q'$; idque ita, ut, si n numerus par fuerit, quo medius seriei terminus constans reddetur, hujus constantis dimidia pars in seriem Q', $=\frac{1}{2}Q$, recipiatur.